Math 72 9.2 Exponential Functions.

Objectives

1) Evaluate exponential functions
2) Graph exponential functions
3) Solve (rigged) exponential functions by the "same base" method.

4) Recognize e = natural base
5) Use compound interest formulas
   a) compounded n times per year
   b) compounded continuously

6) Re-visit Percent increase and percent decrease from chapter 2.

7) Exponential Growth equations and exponential decay functions

Exponential Functions, Exponential Equations, and Word Problems

Objectives:

1. Graph exponential functions.
2. Solve exponential equations algebraically.
3. Solve word problems that result in exponential equations.

1) Evaluate $f(x) = 2^x$ to complete the following table:

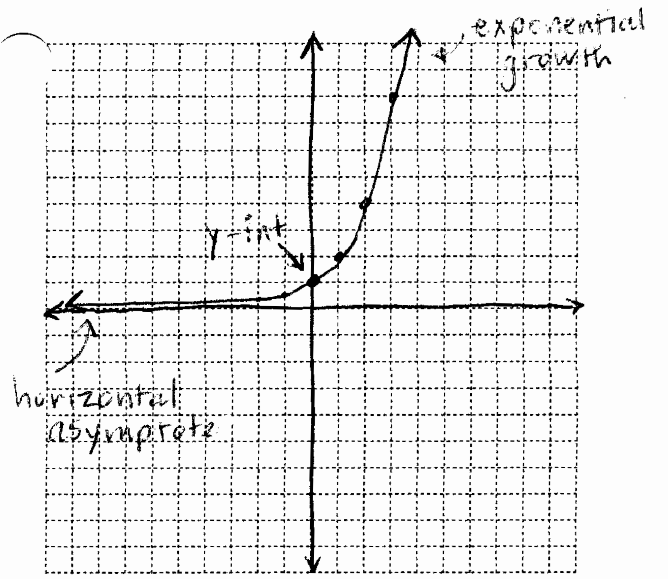
x	$f(x) = 2^x$
-4	$1/16$
-3	$1/8$
-2	$1/4$
-1	$1/2$
0	1
1	2
2	4
3	8
4	16

$2^{-4} = \frac{1}{2^4} = \frac{1}{16}$

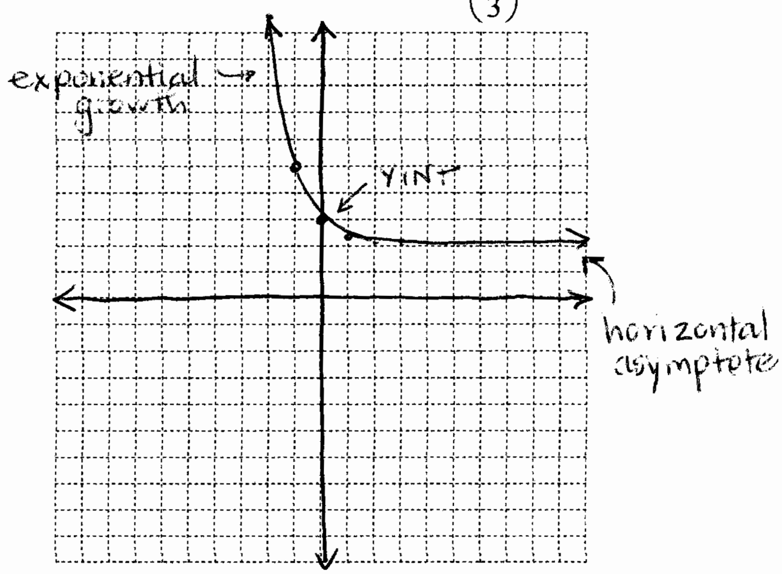
$2^0 = 1$

x	$f(x) = \left(\frac{1}{3}\right)^x + 2$
−4	83
−3	29
−2	11
−1	5
0	3
1	$2\frac{1}{3}$
2	$2\frac{1}{9}$
3	$2\frac{1}{27}$
4	$2\frac{1}{81}$

2) Sketch the graph of $f(x) = 2^x$



3) Sketch the graph of $f(x) = \left(\frac{1}{3}\right)^x + 2$



When sketching the graphs of exponential functions:

1. Plot enough points to show exponential growth. (extend to edge)
2. Plot the y-intercept
3. Plot enough points to show the horizontal asymptote. (extend to edge)

CAUTION: DON'T CHANGE THE SCALE TO SOMETHING > 1 OR YOUR Y-INT WILL BE UNCLEAR

Solve the equation.

step 1: isolate exponential (base & exponent)
step 2: rewrite with same base, simplify exponents.
step 3: ignore bases, set exponents equal.
[More detailed solutions at end of notes]

4) $2^x = 16$ $\quad 2^x = 2^4 \quad \boxed{x=4}$

5) $3 \cdot 2^x = 48$ $\quad 2^x = \frac{48}{3} \Rightarrow 2^x = 16 \Rightarrow 2^x = 2^4 \Rightarrow \boxed{x=4}$

6) $2^{3x+1} = 16$ $\quad 2^{3x+1} = 2^4 \Rightarrow 3x+1 = 4 \Rightarrow 3x = 3 \Rightarrow \boxed{x=1}$

7) $2 \cdot 9^x = 54$ $\quad 9^x = \frac{54}{2} \Rightarrow 9^x = 27 \Rightarrow (3^2)^x = 3^3 \Rightarrow 3^{2x} = 3^3 \Rightarrow 2x = 3 \Rightarrow \boxed{x = \frac{3}{2}}$

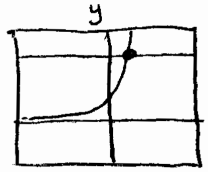
8) $4^{x+3} = 8^x$ $\quad (2^2)^{x+3} = (2^3)^x \Rightarrow 2^{2x+6} = 2^{3x} \Rightarrow 2x+6 = 3x \Rightarrow \boxed{x=6}$

9) $\frac{5}{4} = 5 \cdot 2^{3x}$ $\quad \Rightarrow \frac{5}{4} \cdot \frac{1}{5} = 2^{3x} \Rightarrow \frac{1}{4} = 2^{3x} \Rightarrow 2^{-2} = 2^{3x} \Rightarrow -2 = 3x \Rightarrow \boxed{x = \frac{-2}{3}}$

exponential on RHS!

10) Use GC to estimate the solution to $5^x = 10$

$y_1 = 5$^X $\quad$ WINDOW $\quad$ CALC
$y_2 = 10$ $\quad$ YMAX=15 $\quad$ 2nd TRACE $\quad$ 5. Intersect



$\boxed{x \approx 1.43}$

11) Find the amount owed at the end of 5 years if \$3000 is loaned at a rate of 10% compounded quarterly.

compound interest formula.

$A = P\left(1 + \frac{r}{n}\right)^{nt}$

A = ?
P = 3000
r = .10
n = 4 times per year
t = 5 years

$A = 3000\left(1 + \frac{.1}{4}\right)^{(4 \cdot 5)}$

$A \approx 4915.849$

$\boxed{\$4915.85}$

12) An unusually wet spring has caused the size of the Cape Cod mosquito population to increase by 8% each day. If an estimated 200,000 mosquitoes are on Cape Cod on May 12, find how many mosquitoes will inhabit the Cape on May 25. Use $y = 200{,}000(2.7)^{0.08t}$. Round to the nearest thousand.

t = number of days.

May 25
− May 12
13 days

substitute t = 13

$y = 200000(2.7)^{(0.08 * 13)}$

$y = 561{,}886.1$

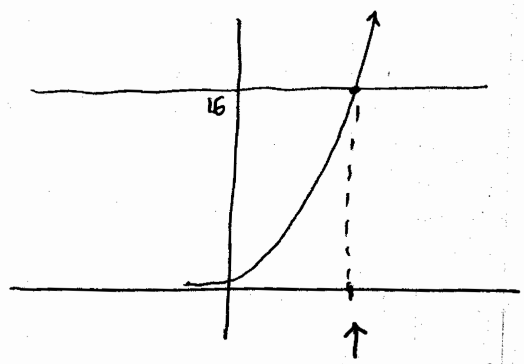
⇑ thousands place

$y = \boxed{562{,}000 \text{ mosquitos}}$

Recall: x^2 exponent 2, base x = "quadratic"
This is a type of polynomial.
Quadratic equations can be solved by factoring, CTS, QF.

2^x exponent x, base 2 = "exponential"
This is not a polynomial, and cannot be solved by factoring or CTS or QF.
Need an entirely new method.

✓ (4) Solve $2^x = 16$
a) using GC
b) using algebra



This x-value is the solution.
$x=4$

a) GC $y_1 = 2 \wedge X$
WINDOW YMAX = 20
$y_2 = 16$

$\boxed{\text{2nd}}\boxed{\text{TRACE}}$ = CALC
5. Intersect $\boxed{\text{ENTER}}\ \boxed{\text{ENTER}}\ \boxed{\text{ENTER}}$

solution $\boxed{x=4}$
check: $2^4 = 16$.

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b) Algebra.
<u>step 1</u>: Find a number which can be used as **the base** of the exponential on both LHS and RHS.
This is called the <u>common base</u>. It is usually smaller than (or equal to) the given numbers.
Rewrite LHS and RHS using this base.

base 2:
$2^x = 2^4$

<u>step 2</u>: Since the bases are now equal, the exponents must be equal.
$\boxed{x=4}$

✓ 5 Solve $3 \cdot 2^x = 48$

<u>step 0:</u> Isolate the exponential 2^x before finding a common base.

$$\frac{3 \cdot 2^x}{3} = \frac{48}{3}$$ divide both sides by 3

$2^x = 16$ now it's the same problem as 1

$2^x = 2^4$

$\boxed{x=4}$

* Extra $3 \cdot 2^x + 5 = 53$

✓ 6 Solve $2^{3x+1} = 16$

$2^{3x+1} = 2^4$ common base

$3x+1 = 4$ set exponents equal

$3x = 3$ isolate x.

$\boxed{x=1}$

✓ 7 Solve $2 \cdot 9^x = 54$

Isolate exponential

$9^x = 27$

$9^2 = 81$ is too big.

but $9 = 3^2$ and $27 = 3^3$ } so common base is 3.

$(3^2)^x = (3^3)$ substitute equivalent exponential expressions using common base.

$3^{2x} = 3^3$ use exponent laws to simplify $(a^n)^m = a^{n \cdot m}$

$2x = 3$ set exponents equal

$\boxed{x = \frac{3}{2}}$ isolate x

✓ 8 Solve $4^{x+3} = 8^x$

$4^2 = 16$ is too big.

but $2^2 = 4$ and $2^3 = 8$ } so common base is 2.

$(2^2)^{x+3} = (2^3)^x$ substitute equivalent exponential expressions using common base

$2^{2x+6} = 2^{3x}$ use exponent laws to simplify.

$2x+6 = 3x$ set exponents equal

$\boxed{6 = x}$ isolate x

✓ 9 Solve $\frac{5}{4} = 5 \cdot 2^{3x}$

Isolate exponential.
remember that $\frac{1}{n} = n^{-1}$

$\frac{1}{4} = 2^{3x}$

So $\frac{1}{4} = 4^{-1}$

$4^{-1} = 2^{3x}$

while it's not a necessary step to do only this, it can be helpful.

We still need a common base

$\left.\begin{array}{l} 2 = 2^1 \\ 4 = 2^2 \end{array}\right\}$ so 2 is our common base.

$(2^2)^{-1} = 2^{3x}$

$2^{-2} = 2^{3x}$ exponent laws

$-2 = 3x$ set exponents equal

$\boxed{x = -\frac{2}{3}}$ isolate x

Extra

x A Solve $25^x = \frac{1}{5}$

$\left.\begin{array}{l} \frac{1}{5} = 5^{-1} \\ 25 = 5^2 \end{array}\right\}$ common base is 5

$(5^2)^x = 5^{-1}$

$5^{2x} = 5^{-1}$ exponent laws

$2x = -1$ set exponents equal

$\boxed{x = -\frac{1}{2}}$ isolate x.

Extra

x B Solve $4^{3x-7} = 32^{-2}$

$\left.\begin{array}{l} 4^2 = 16 \text{ not quite} \\ 16^2 = 256 \end{array}\right\}$ common base is 2 ⇒ SMALLER

$(2^2)^{3x-7} = (2^5)^{-2}$

$\left\{\begin{array}{l} 4 = 2^2 \\ 32 = 2^5 \end{array}\right.$

$2^{6x-14} = 2^{-10}$ exponent laws

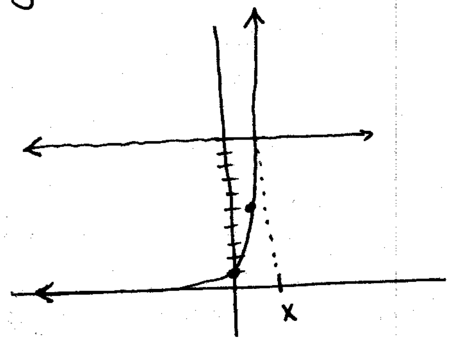
$6x - 14 = -10$ set exponents equal

$6x = 4$ isolate x

$\boxed{x = \frac{2}{3}}$ reduce

(10) Use GC to estimate solution to $5^x = 10$.
Why can't we solve this exactly (yet)?

$y_1 = 5^x$
$y_2 = 10$



$\boxed{\text{WINDOW}}$ YMAX = 12

$\boxed{\text{2nd}}\ \boxed{\text{TRACE}}$ = CALC
5. Intersect $\boxed{\text{ENTER}}\boxed{\text{ENTER}}\boxed{\text{ENTER}}$

$x \approx 1.4306766$

$\boxed{x \approx 1.43}$

This means that $5^1 = 5$ is less than 10
but $5^2 = 25$ is greater than 10.

We need an exponent value which is not a whole number.
To do this exactly, we will use logarithms in 9.8.

Recall:

Compound Interest Formula

$$A = P\left(1 + \frac{r}{n}\right)^{(nt)}$$

A = amount at end including principal, interest and interest on interest.

P = principal or initial amount

r = interest rate, % written as decimal.

n = # times compounded per year

t = time in years.

Notice that this formula is an exponential if A(t) and P, r, n known.

ex: $P = 5000$
$r = 0.05$
$n = 12$

$$A(t) = 5000 \cdot \left(1 + \frac{.05}{12}\right)^{12t}$$

$$A(t) = \underbrace{5000}_{\text{coefficient}} \cdot \underbrace{\left(\frac{241}{240}\right)}_{\text{base}}^{\overbrace{12t}^{\text{exponent.}}}$$

e is called the <u>natural base</u>

e, like π, is an irrational number

- it cannot be written as a fraction of two integers
- its decimal does not terminate
- its decimal does not repeat

- e is a constant, always the same value, not a variable.

e 2.718281828 45 90 45 23 53 60 28 74

π 3.1415926 53 58 97 93 23 84 62 64 33

Approximate to nearest ten-thousandth using GC

(11) e

[2nd] [LN] [1] [ENTER]

$\underbrace{\text{2nd LN}}_{e^\wedge}$ e^1

$= 2.71828$

$\approx \boxed{2.7183}$ ← approximate

e ← exact

e to 25 decimal places

$e \approx 2.7182818284590452353603\ldots$

Δs

Extra

no (C) e^2

[2nd] [LN] [2] [ENTER]

$= 7.38905$

$\approx \boxed{7.3891}$ ← approximate

e^2 ← exact

e is an irrational number!

- its decimal does not terminate
- Its decimal does not repeat
- if we need an exact answer, we will write an expression using e.

no (D) e^3

$= 20.08553$

$\approx \boxed{20.0855}$ ← approximate

e^3 ← exact

✓ (12) e^{-1}

$= .36787$

$\approx \boxed{.3679}$ ← approximate

e^{-1} ← exact

Solve.

✓ (13) $e^x = e^3$ — same base

$\boxed{x=3}$ — Set exponents equal

(14) $e^4 = e^x \cdot e^x$ — exponent laws

✓ $e^4 = e^{x+x}$

$e^4 = e^{2x}$

$4 = 2x$

$\boxed{2 = x}$

A simpler question

$x^2 \cdot x^3 = x^{2+3}$

add exponents

Compound Interest Formula (n times per year)

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

A = amount in account
P = principal
r = %, interest rate, written as decimal
n = # times compounded each year
t = time in years

What if we want to compound continuously?
We calculate interest and add it, and do it again....
We get a computer to do this without stopping.
This means n = really big number (in a practical sense)
or $n = +\infty$ (in a theoretical sense).

If we choose $P=1$, $r=1$, and $t=1$ so we can see only the effect of n on our formula, we get....

$$A = 1\left(1+\frac{1}{n}\right)^{n\cdot 1}$$

$$A = \left(1+\frac{1}{n}\right)^{n}$$

Let's evaluate this for values of n that get bigger & bigger.

GC : $\boxed{Y=}$
$Y_1 = (1 + 1/X)$ ^ X

TBLSET
$\boxed{2nd}$ $\boxed{WINDOW}$ Independent $\boxed{Ask}$

TABLE
$\boxed{2nd}$ $\boxed{GRAPH}$

X	Y_1
100	2.7048138
1000	2.7169238
10000	2.7169239
100000	2.7181459
1000000	2.7182805
10000000	2.7182817

Use ▷ to move over Y_1 values. Extra decimal places are then displayed at bottom of screen.

Woah! It looks like $e \approx 2.718281828...!$

This gives us a different formula for

<u>Continuously Compounded Interest</u>

$A = Pe^{rt}$

$A =$ amount in account
$P =$ principal
$r =$ %, interest rate, written as decimal
$t =$ time in years

(15) Find the amount in account if \$3000 is invested for 5 years at 4% interest compounded
a) monthly
b) continuously

a) $A = P\left(1+\frac{r}{n}\right)^{nt}$

$A = ?$
$P = 3000$
$r = .04$ (4%)
$n = 12$ times per year
$t = 5$ years

$A = 3000\left(1+\frac{.04}{12}\right)^{12\cdot 5}$

$= \boxed{\$3662.99}$

3662.989 rounded to nearest cent

b) $A = Pe^{rt}$

$A = ?$
$P = 3000$
$r = .04$
$t = 5$

$A = 3000e^{(.04)(5)}$

$= \boxed{\$3664.21}$

3664.208 rounded to nearest cent

Math 72

Remember Percent Increase and Percent Decrease?

New Amount = Base + %·Base (increase)
New Amount = Base − %·Base (decrease)

If we factor out "Base", we get

New Amount = Base (1 + %) (increase)
New Amount = Base (1 − %) (decrease)

Let's call the Base = C , New Amount = y
and % = r (for rate) * write as a decimal *

$$y = C(1+r) \quad \text{(increase)}$$
$$y = C(1-r) \quad \text{(decrease)}$$

(16) A city with population 350,000 has been growing at a rate of 9% per year. Assume this rate continues and find
a) population after 1 year
b) population after 2 years
c) population after 19 years
d) population y after t years

a) $y = 350{,}000(1+.09)$
$y = \boxed{381{,}500 \text{ people}}$ after 1yr

b) $y = 381{,}500(1+.09) = 350{,}000(1+.09)(1+.09)$
$y = \boxed{415835 \text{ people}}$ after 2yr

c) Notice that in part a, we multiplied by 1.09.
in part b, we multiplied by $(1.09)^2$

$y = 350{,}000(1.09)^{19}$
$y = \boxed{1{,}799{,}581 \text{ people}}$ after 19 yr

d) $\boxed{y = 350000(1.09)^t}$ ← This is called an exponential growth function.

17 Suppose a nearby city with population 350,000 is decreasing by 9% per year. Assume this trend continues and find

a) population after 19 years
b) population after t years

a) $y = 350000(1 - .09)^{19}$

$= 5832.496656$

$\approx \boxed{5832 \text{ people}}$

b) $y = 350000(1 - .09)^{t}$

$\boxed{y = 350000(.91)^{t}}$ ← This is called an <u>exponential decay function</u>.

<u>Exponential Growth Function</u>

$y = C(1+r)^t$

C = starting amount
r = % increase for each unit of time
t = # units of time

<u>Exponential Decay Function</u>

$y = C(1-r)^t$

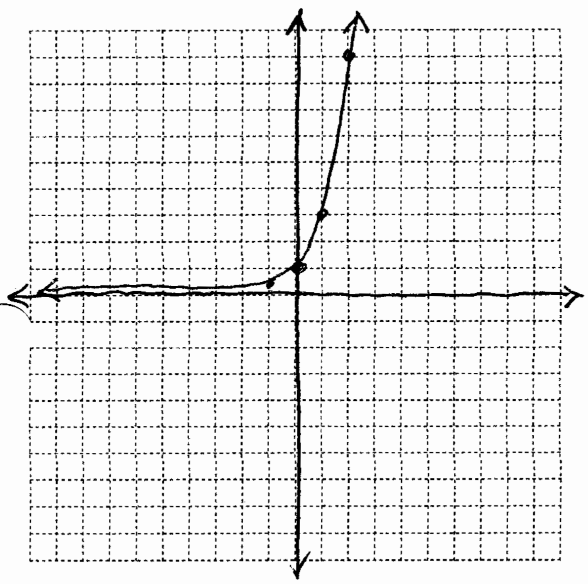
C = starting amount
r = % decrease for each unit of time
t = # units of time

EXTRAS:

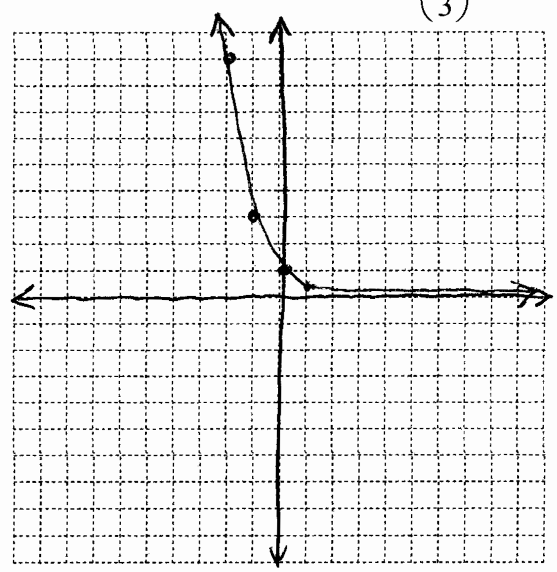
13) Evaluate $f(x) = 3^x$ to complete the following table:

x	$f(x) = 3^x$
-4	1/81
-3	1/27
-2	1/9
-1	1/3
0	1
1	3
2	9
3	27
4	81

14) Sketch the graph of $f(x) = 3^x$

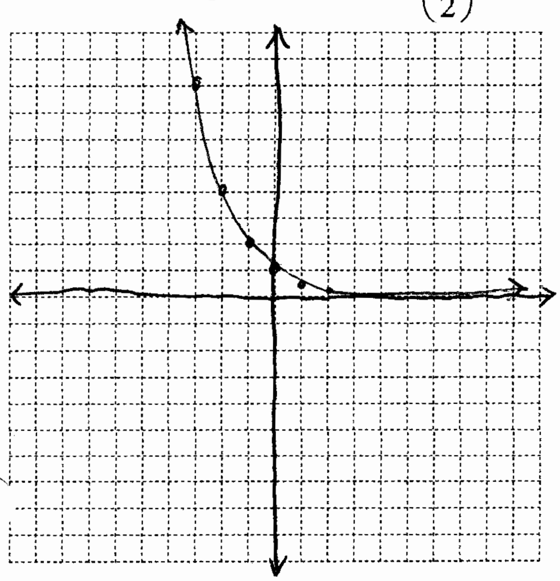


15) Sketch the graph of $f(x) = \left(\frac{1}{3}\right)^x$



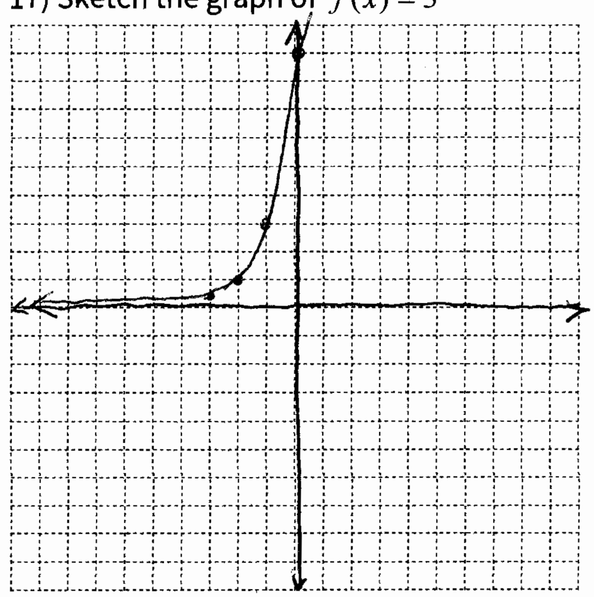
x	$(\frac{1}{3})^x$
-4	81
-3	27
-2	9
-1	3
0	1
1	1/3
2	1/9
3	1/27
4	1/81

16) Sketch the graph of $f(x) = \left(\frac{1}{2}\right)^x$



x	f(x)
-4	16
-3	8
-2	4
-1	2
0	1
1	1/2
2	1/4
3	1/8
4	1/16

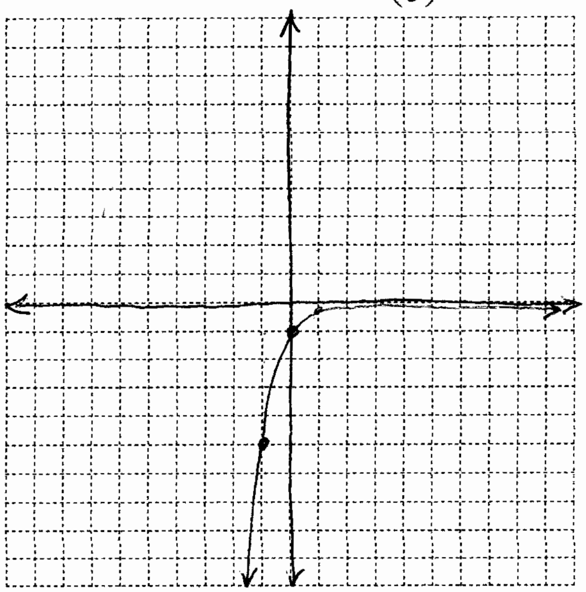
17) Sketch the graph of $f(x) = 3^{x+2}$



x	f(x)
-4	1/9
-3	1/3
-2	1
-1	3
0	9
1	27
2	81
3	way too big
4	

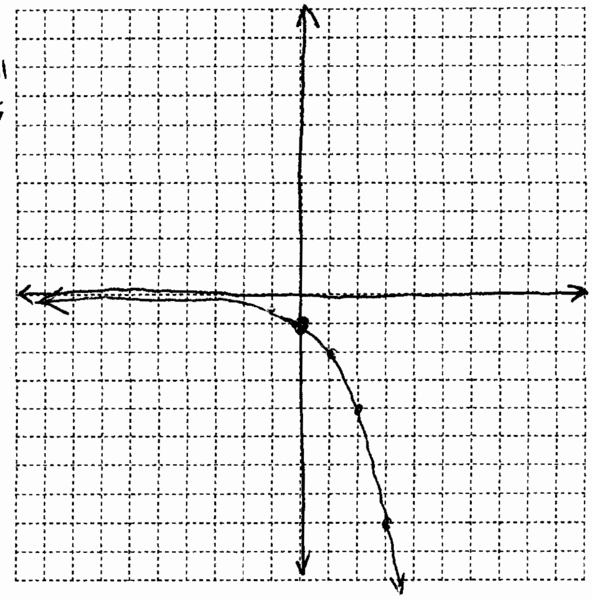
exponent before multiplying

18) Sketch the graph of $f(x) = -\left(\frac{1}{5}\right)^x$



x	f(x)
−4	too small
−3	−125
−2	−25
−1	−5
0	−1
1	−1/5
2	−1/25
3	−1/125
4	too close to 0

19) Sketch the graph of $f(x) = -2^x$



x	f(x)
−4	−1/16
−3	−1/8
−2	−1/4
−1	−1/2
0	−1
1	−2
2	−4
3	−8
4	−16

Solve the equation.

20) $25^x = \frac{1}{5} \Rightarrow (5^2)^x = 5^{-1} \Rightarrow 5^{2x} = 5^{-1} \Rightarrow 2x = -1$ $\boxed{x = -\frac{1}{2}}$

21) $4^{3x-7} = 32^{-2} \Rightarrow (2^2)^{3x-7} = (2^5)^{-2} \Rightarrow 2^{6x-14} = 2^{-10}$
$6x - 14 = -10 \Rightarrow 6x = 4$ $\boxed{x = 2/3}$

22) As a result of the Chernobyl nuclear accident, radioactive debris was carried through the atmosphere, affecting cows and the milk they provide. The percent *y* of radioactive material in raw milk after *t* days is estimated by $y = 100(2.7)^{-0.1t}$. Find the expected percent of radioactive material in the milk after the Chernobyl accident. Give answers rounded to the nearest tenth of a percent.

a. 0 days $y = 100(2.7)^0 \Rightarrow 100\%$

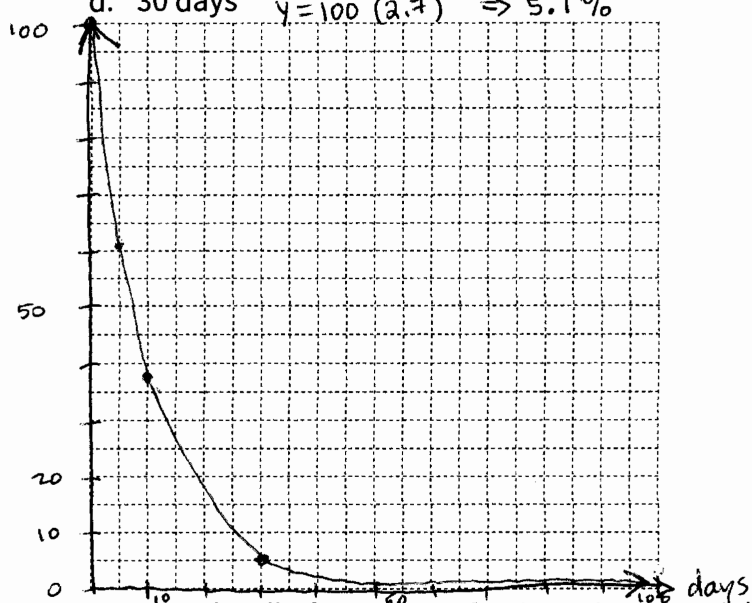
b. 1 day $y = 100(2.7)^{-0.1} \Rightarrow 90.5\%$

c. 10 days $y = 100(2.7)^{((-0.1)(10))} \Rightarrow 37.0\%$

d. 30 days $y = 100(2.7)^{-3} \Rightarrow 5.1\%$

e. 100 days $y = 100(2.7)^{-10} \Rightarrow 0.0\%$ (rounded)

a. Make a table of additional points and sketch a graph of $y = 100(2.7)^{-0.1t}$



x	y
5	60.9%
50	0.7%
75	0.1%

23) An accidental spill of 75 grams of radioactive material in a local stream has led to the presence of radioactive debris decaying at a rate of 4% each day. Find how much debris still remains after 14 days. Use $y = 75(2.7)^{-0.04t}$ Round to the nearest tenth of a gram. $y = 75(2.7)^{(-.04 * 14)} = \boxed{43.0 \text{ grams}}$